The Three-Equation Model

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1 Introduction

This note describes the 'Three-Equation Model', a model of the conditions in a thin layer of seawater in direct contact with ice. According to Jenkins et al., (2001), the model was first described in Josberger, (1983). A more detailed description of the model and a comparison with the one and two-equation variations is given in Holland and Jenkins (1999).

The model parameterizes the conditions in a boundary layer at the ocean-ice interface where differences in the molecular diffusivity of heat and salt are important. Conditions close to the ocean-ice interface may be distinct from those of the "far-field" ocean, where eddy mixing dominates over molecular diffusion. In numerical models, the far-field properties are typically considered to be the properties of the ocean model grid cell in closest contact with the ice.

Consider the boundary layer of seawater of some small fixed thickness in contact with ice. Within the boundary layer we assume that seawater is in thermal equilibrium with the ice at the an in situ seawater freezing point $T = T_B$, which is a function of salinity $S = S_B$ and local pressure p.

1.1 Equation 1: Freezing Point

The first equation of the three equations is a linearization of the seawater freezing point as a function of salinity and pressure,

$$T_B = a_0 S_B + c_0 + b_0 p \tag{1}$$

1.2 Equation 2: Heat Balance at Ice/Ocean Interface

A heat balance equation at the ice/ocean interface constitutes the second equation,

$$\rho_w \, c_{pw} \langle w' T' \rangle + \rho_I c_{pI} \kappa_I \left. \frac{\partial T_I}{\partial z} \right|_B = -Lq \tag{2}$$

The sum of turbulent ocean heat flux to the interface, $\langle w'T' \rangle$ and thermal conductive heat flux out of the interface (towards the surface) is proportional to ice mass flux, q (with $q < 0 \rightarrow$ melting).

Conductive heat fluxes are a function of thermal diffusivity, κ ($m^2 s^{-1}$), and the ice temperature gradient at the boundary, $\partial_z T_I|_B$. L is the latent heat of fusion ($J\,kg^{-1}$). Units of the ice mass flux, q are ($kg\,s^{-1}\,m^{-2}$). c_{pw} is the seawater heat capacity, units $J\,kg^{-1}\,K^{-1}$. All terms in the second equation are $W\,m^{-2}$. Turbulent heat fluxes are parameterized using as a turbulent exchange coefficient, γ_T (ms^{-1}) which acts on the difference between the far field ocean temperature, T_O, and the temperature within the boundary layer, T_B, as

$$\rho_w\, c_{pw} \gamma_T \left(T_O - T_B\right) + \rho_I c_{pI} \kappa_I \left.\frac{\partial T_I}{\partial z}\right|_B + Lq = 0 \tag{3}$$

In the case of a linear temperature gradient throughout the ice between the relatively warm ice-ocean interface (where $T = T_B$) and the generally colder surface or interior (where $T = T_B$) over a distance D_I, then the heat balance equation becomes,

$$\underbrace{\rho_w\, c_{pw} \gamma_T \left(T_O - T_B\right)}_{>0} + \underbrace{\rho_I c_{pI} \kappa_I \frac{(T_I - T_B)}{D_I}}_{<0} + \underbrace{Lq}_{<0} = 0 \tag{4}$$

The residual of heat flux convergence and conduction results in ice melting or freezing.

1.3 Equation 3: Salt Balance in the Boundary Layer

A salt balance equation in the layer constitutes the last of the three equations,

$$\langle w'S' \rangle + w_B S_B - w_B S_I = 0 \tag{5}$$

$$\langle w'S' \rangle + w_B (S_B - S_I) = 0 \tag{6}$$

Turbulent salt fluxes, $\langle w'S' \rangle$, *into* the boundary layer are offset by the advective flux of salt *out* the boundary layer, $w_B S_B$, and the advective flux of salt from melted ice *into* the boundary layer, $-w_B S_I$. Note: the velocity of water into and out of the boundary layer, w_B must must be identical for mass continuity. Pure meteoric ice has zero salinity so $S_I = 0$. Negative velocities are 'downward'.

Like heat fluxes, the turbulent salt flux from the far-field ocean layer ($S = S_O$) into the boundary layer ($S = S_B$) can be parameterized with an exchange coefficient, $\gamma_S\,[ms^{-1}]$:

$$\gamma_S \left(S_O - S_B\right) + w_B (S_B - S_I) = 0 \tag{7}$$

Terms here have units $psu\, ms^{-1}$.

The ice mass flux, q $[kgs^{-1}m^{-2}]$, can be expressed with respect to the density and (downward) velocity of waters moving through the boundary layer as,

$$q = \rho_w w_B \tag{8}$$

Where ρ_w is the density of water in boundary layer. Negative q denotes melting ice and negative w_B denotes downward velocity. In Holland and Jenkins, ρ_w is a reference density of 1025 kgm^{-3}. Perhaps a more natural way of expressing the ice mass flux would be with respect to an ice melt rate, w_I $[ms^{-1}]$, with negative w_I denoting melting, positive denoting growth:

$$q = \rho_I w_I \tag{9}$$

Which leads to the equivalency,

$$w_I = \frac{\rho_w}{\rho_I} w_B \tag{10}$$

One may express Eq 7 in terms of q by multiplying through by ρ_w:

$$\underbrace{\rho_w \gamma_S (S_O - S_B)}_{>0} + \underbrace{q (S_B - S_I)}_{<0} = 0 \tag{11}$$

1.4 3 equation summary

$$T_B = a_0 S_B + c_0 + b_0 p \tag{12}$$

$$\rho_w c_{pw} \gamma_T (T_O - T_B) + \rho_I c_{pI} \kappa_I \frac{(T_I - T_B)}{D_I} + Lq = 0 \tag{13}$$

$$\rho_w \gamma_S (S_O - S_B) + q S_B = 0 \tag{14}$$

2 Solution Strategy

The three equation model can be reduced to a quadratic equation for S_B. With S_B known, one can immediately solve for the other two unknowns, T_B, and q. The rest of this section describes the steps requires to form the quadratic equation.

Make the following definitions (labeled to be as consistent as possible with those used by M. Losch *shelfice* MITgcm package.

$$\begin{aligned}
\epsilon_1 &= \rho_w c_{pw} \gamma_T \\
\epsilon_2 &= \rho_w \gamma_S L \\
\epsilon_3 &= \rho_I c_{pI} \kappa / D_I \\
\epsilon_4 &= c_0 + b_0 p \\
\epsilon_6 &= \epsilon_4 - T_O \\
\epsilon_7 &= \epsilon_4 - T_I \\
A &= a_0(\epsilon_1 + \epsilon_3) \\
B &= \epsilon_1 \epsilon_6 + \epsilon_3 \epsilon_7 - \epsilon_2 - S_I A \\
C &= \epsilon_2 S_O - S_I(\epsilon_1 \epsilon_6 + \epsilon_3 \epsilon_7)
\end{aligned}$$

First, reduce the three equations to two by substituting T_B from Eq 1 into Eq 4 and use the above definitions for ϵ_1, ϵ_3, and ϵ_4,

$$\epsilon_1 \left(T_O - [a_0 S_B + \epsilon_4]\right) + \epsilon_3 (T_I - [a_0 S_B + \epsilon_4]) + Lq = 0 \tag{15}$$

Simplify Eq 15 by using ϵ_6 for $\epsilon_4 - T_O$ and ϵ_7 for $\epsilon_4 - T_I$. Also multiply by sides by -1.

$$\epsilon_1 \left(a_0 S_B + \epsilon_6\right) + \epsilon_3 (a_0 S_B + \epsilon_7) - Lq = 0 \tag{16}$$

Now solve Eq 11 for q,

$$q = -\rho_w \gamma_S \left(S_O - S_B\right) \left(S_B - S_I\right)^{-1} \tag{17}$$

substitute q from Eq 17 into Eq 16, making use of ϵ_2:

$$\epsilon_1 \left(a_0 S_B + \epsilon_6\right) + \epsilon_3 (a_0 S_B + \epsilon_7) + \epsilon_2 \left(S_O - S_B\right) \left(S_B - S_I\right)^{-1} = 0 \tag{18}$$

Multiply both sides by $\left(S_B - S_I\right)^{-1}$ and collect terms,

$$\left(S_B - S_I\right) \left[S_B \left(a_0 (\epsilon_1 + \epsilon_3) + \epsilon_1 \epsilon_6 + \epsilon_3 \epsilon_7\right)\right] + \epsilon_2 \left(S_O - S_B\right) = 0 \tag{19}$$

Multiply through by $(S_B - S_I)$, collect terms in S_B^n, and make use of A to form the quadratic equation.

$$S_B^2 A + S_B \left(\epsilon_1 \epsilon_6 + \epsilon_3 \epsilon_7 - \epsilon_2 - S_I A\right) + \epsilon_2 S_O - S_I (\epsilon_1 \epsilon_6 + \epsilon_3 \epsilon_7) = 0 \tag{20}$$

Substitute B and C in Eq 20,

$$S_B^2 A + S_B B + C = 0 \tag{21}$$

Recall that solution of this quadratic equation is given by the quadratic formula,

$$S_B = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{22}$$

From which one chooses the positive root of S_B.

With the appropriate root of S_B, q is given directly by Eq 17 and T_B is given through Eq 1.

3 Oddities

3.1 Alternative formulation of B when beneath an ice shelf cavity

If the magnitude of the thermal conduction distance, D_I is approximately the same as the pressure (in decibars) where the calculations are being made, $D_I \approx p$, then the term, $\epsilon_3 \epsilon_7$ can be reduced to,

$$\epsilon_3 \epsilon_7 = \frac{\rho_I c_{pI} \kappa_I}{D_I} (\epsilon_4 - T_I) \tag{23}$$

$$= \frac{\rho_I c_{pI} \kappa_I}{D_I} (c_0 + b_0 p - T_I) \tag{24}$$

$$= \rho_I c_{pI} \kappa_I \left[\frac{(c_0 - T_I)}{D_I} - b_0\right] \tag{25}$$

In Martin's *shelfice* package an intermediate epsilon variable is defined: $\epsilon_{3a} = \rho_I c_{pI} \kappa_I$. Substituting ϵ_{3a} into the above expression to,

$$\epsilon_3 \epsilon_7 = \epsilon_{3a} \left[\frac{(c_0 - T_I)}{D_I} - b_0 \right] \tag{26}$$

the form found in the routine.

3.2 Ice with Zero Salinity

If ice has zero salinity, $S_I = 0$, then the B and C coefficients of the quadratic equation become

$$B = \epsilon_1 \epsilon_6 + \epsilon_3 \epsilon_7 - \epsilon_2 \tag{27}$$

$$C = \epsilon_2 S_O \tag{28}$$

Which, when combined with the expression for $\epsilon_3 \epsilon_7$ in Eq 26 are found in one formulation of the *shelfice* package. Note that since D_I the *shelfice* package is the draft of the floating ice and not the ice shelf thickness, the ice temperature (T_I) used in the conductivity term should be considered the temperature at sea level.

3.3 Nonlinear Temperature Gradient Through the Ice Shelf

According to the Holland and Jenkins (1999) a nonlinear temperature profile within the ice can develop as a result of ice advection (mass convergence) and melted ice at the ice-ocean boundary. Consider an ice shelf with an initially linear ice shelf temperature gradient ($\partial_z z T_I = 0$) between T_s at the surface and T_B at the ice-ocean interface with $T_I \leq T_B$. Assuming the underside of the ice shelf is flat and that the geometry of the cavity is fixed, ice mass loss through basal melting is exactly replaced by ice mass convergence via advection from above. If the rate at which colder ice from above advances (w_I) is faster than the rate at which thermal conduction can re-equilibrate the linear temperature gradient then a nonlinear temperature profile will develop (see HJ Sec. 3). When new ice is accumulating at the ice-ocean interface, the temperature of the new ice is assumed to be T_b and therefore the ice temperature gradient at the boundary is approximately zero ($\partial_z T_I \approx 0$).

As described in Jenkins et al., (2001), parameterizing this effect can be accomplished by modifying the temperature gradient at the ice interface, Eq 3, by a scalar factor, Π, as

$$\left. \frac{\partial T_I}{\partial z} \right|_B = \Pi \frac{T_S - T_B}{D_I} \tag{29}$$

with

$$\Pi = \left\{ \begin{array}{ll} \dfrac{w_I \, D_I}{\kappa_I} & \text{melting, } w_I > 0 \\ 0 & \text{freezing, } w_I < 0 \end{array} \right\} \tag{30}$$

Using this alternative expression for the temperature gradient at the ice-ocean boundary in Eq 29 one finds,

$$\rho_w c_{pw} \kappa \left.\frac{\partial T_I}{\partial z}\right|_B = \rho_I c_{pI} \kappa \Pi \frac{(T_B - T_I)}{D_I} = \rho_I c_{pI} w_I (T_B - T_I) \tag{31}$$

An expression for the ice velocity, w_I, can be found using the meltwater mass flux, q, as

$$w_I = \frac{q}{\rho_I} \tag{32}$$

Define a switch variable, ψ, which is one if $w_I > 0$ and zero otherwise. Replacing the temperature conduction term from Eq 31 into the heat balance equation Eq 3 and using the ψ variable to account for the switch in Π from Eq 30 yields,

$$\rho_w \, c_{pw} \gamma_T \, (T_O - T_B) - \psi q c_{pI} (T_B - T_I) - Lq = 0 \tag{33}$$

The algebra proceeds along a similar track as before with the main exception that the the meltwater mass flux q now appears in two locations in the above expression. In the end a quadratic equation is again found with different constant coefficients,

$$\begin{aligned}
\epsilon_8 &= \rho_w \gamma_S c_{pI} \\
A &= a_0(\epsilon_1 - \psi \epsilon_8) \\
B &= \epsilon_1 \epsilon_6 + \psi(S_O \epsilon_8 a_0 - \epsilon_8 \epsilon_7) - \epsilon_2 - S_I \epsilon_1 a0 \\
C &= S_O(\psi \epsilon_8 \epsilon_7 + \epsilon_2) - S_I \epsilon_1
\end{aligned}$$

Including this parameterization to account for nonlinear ice T profiles has the main effect of reducing q when melt rates are high because of the increased thermal conduction through the ice due to steepening of local temperature gradients through the ice.

If the solution to the quadratic equation shows freezing conditions, then S_b must be solved for a second time using $\psi = 0$.

4 Some plots

The following pages show the solution to the three-equation model for z = 1 m (Figs 1, 2, 3) and z = 1000 m (Figs 4, 5, 6) at a range of far-field ocean T between -4 and 2 Celsius and S=34.7. The three figures in each depth category correspond to different treatments in ice conduction: no conduction (Figs 1 and 4), linear T profile (Figs 2 and 5), and nonlinear T profile (3 and 6).

The experiments with no conduction and linear conduction are similar for warm far-field ocean T because as the far-field ocean T increases, the boundary layer salinity approaches 0 and T_b asymptotically approaches the pressure-dependent melting point for S=0. As T_b approaches this limit so does the top-bottom ice temperature gradient and therefore the ice conductive heat flux while the turbulent heat flux increases linearly.

When far-field ocean T is colder than then local freezing point, the no ice conduction experiments yield identical answers to the nonlinear ice T profile because $\Pi = 0$ (see Eq 30). In contrast, under freezing conditions heat continues to be conducted out the of ocean through the ice in the linear ice T profile case because in these experiments $T_B < T_I$.

At z=1000 m the conductive heat fluxes in the linear T ice profile are extremely small ($0.054 W/m^2$), essentially zero. In contrast, when a nonlinear ice T profile is considered ice conductive fluxes increase and remains about 10% of the turbulent ocean-ice heat fluxes.

Parameters used in the following plots:

$$\begin{aligned}
a0 &= -0.0575 && C/psu \\
b0 &= -7.61 \times 10^{-4} && C/db \\
c0 &= 0.0901 && \\
T_I &= -20 && C \\
\rho_I &= 917 && kg/m^3 \\
\rho_w &= 1027 && kg/m^3 \\
c_{pI} &= 2000.0 && J/kgK \\
c_{pw} &= 3994 && J/kgK \\
L &= 334 \times 10^3 && J/kg \\
\kappa &= 1.54 \times 10^{-6} && m^2/s \\
\gamma_T &= 1.0 \times 10^{-4} && m/s \\
\gamma_S &= 5.05 \times 10^{-3} * \gamma_T && m/s
\end{aligned}$$

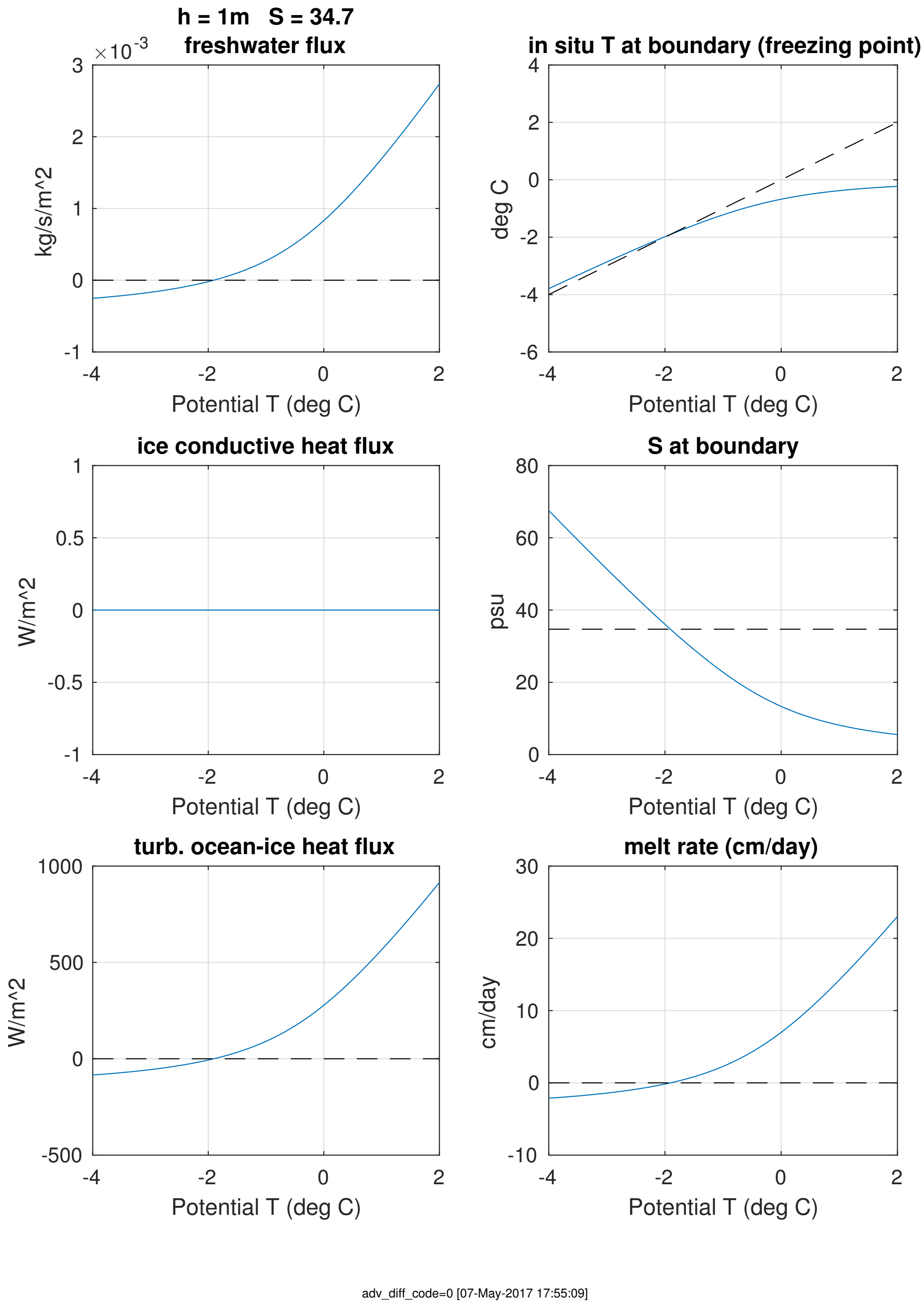

Figure 1: Solution to the Three-Equation model for z = 1 m, no ice conduction. Note that negative freshwater flux corresponds to freezing conditions (in situ $T \leq$ local freezing point).

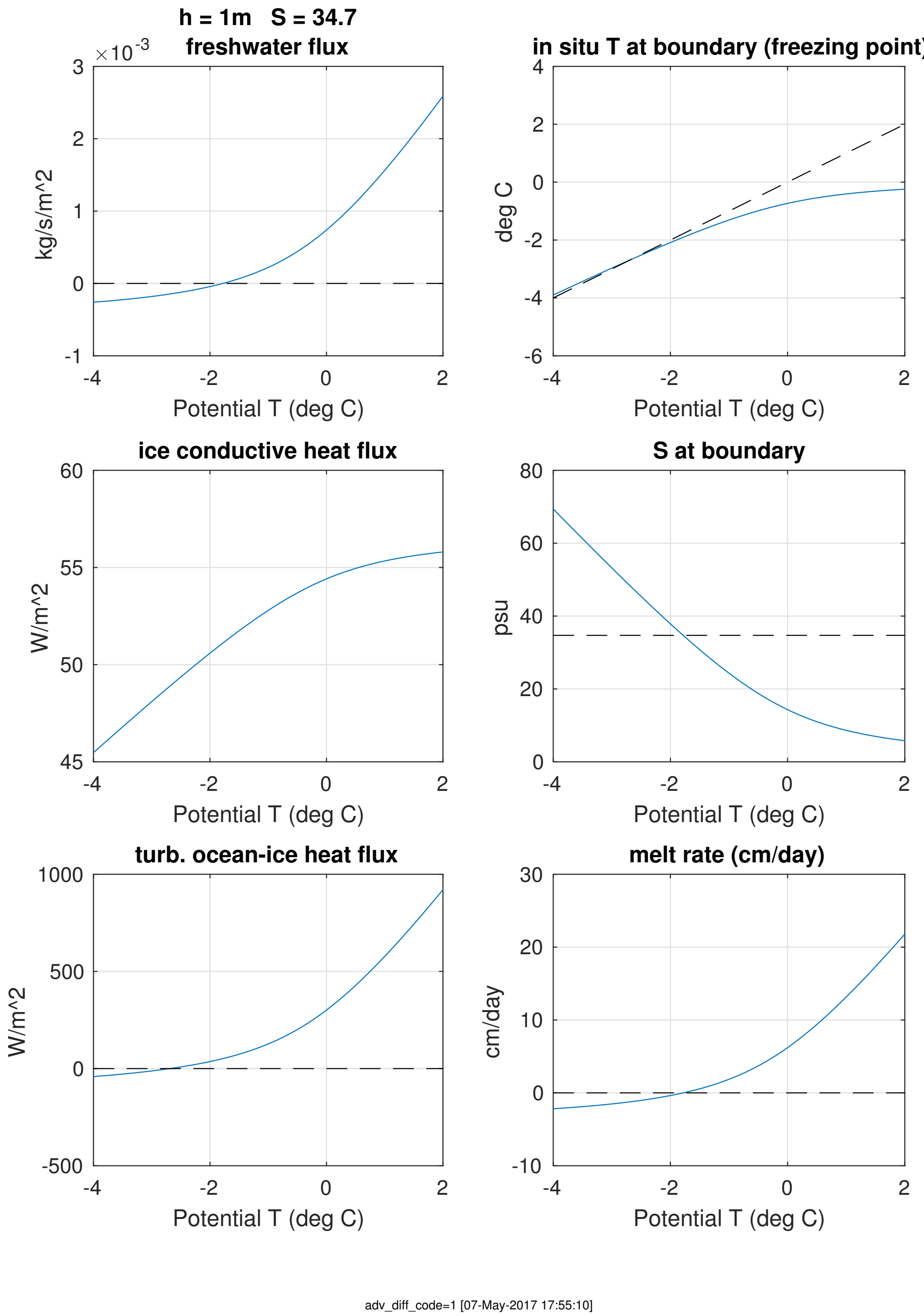

Figure 2: Solution to the Three-Equation model for z = 1 m, ice conduction with linear ice T profile

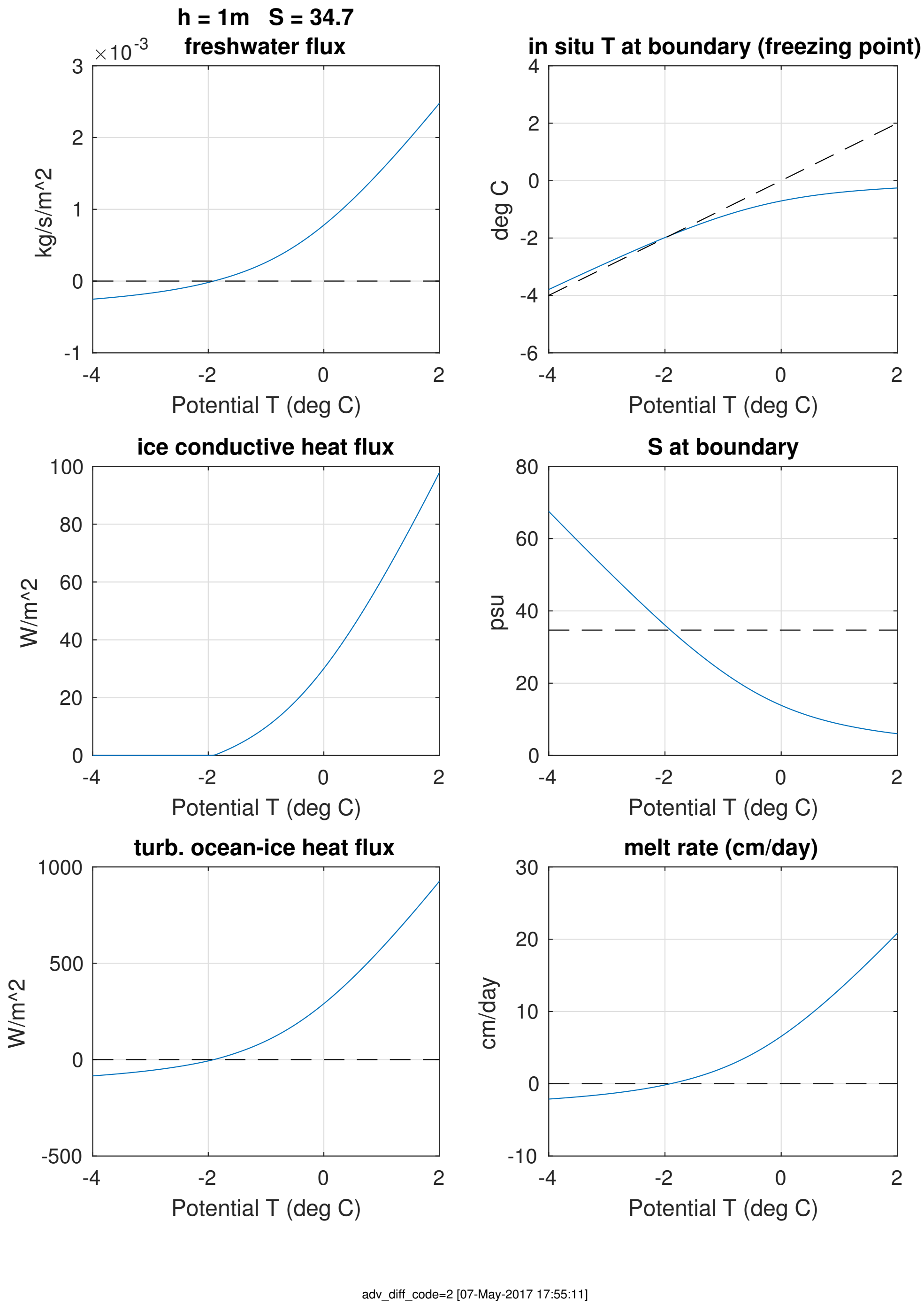

Figure 3: Solution to the Three-Equation model for z = 1 m, ice conduction with nonlinear ice T profile

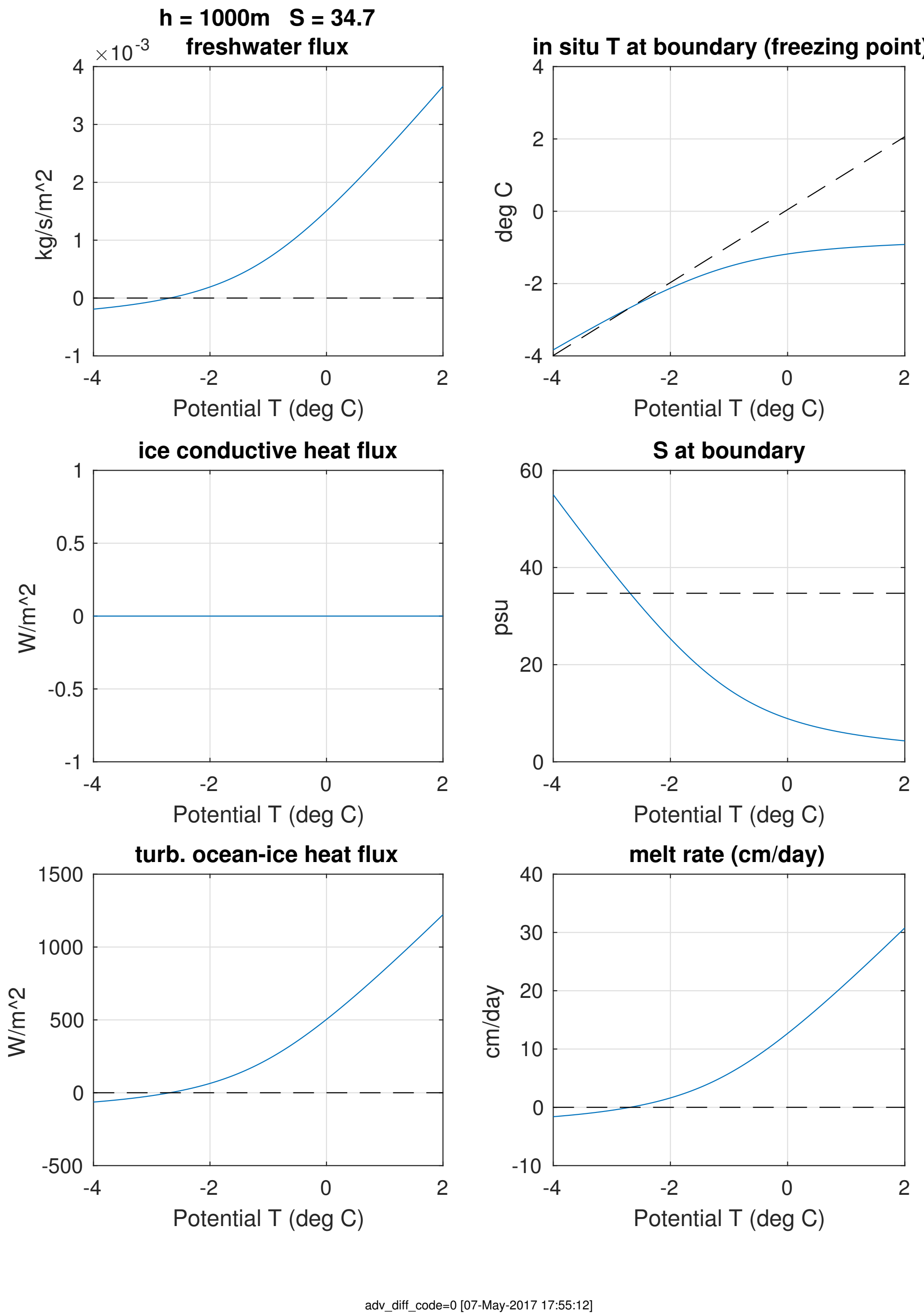

Figure 4: Solution to the Three-Equation model for z = 1000 m, no ice conduction

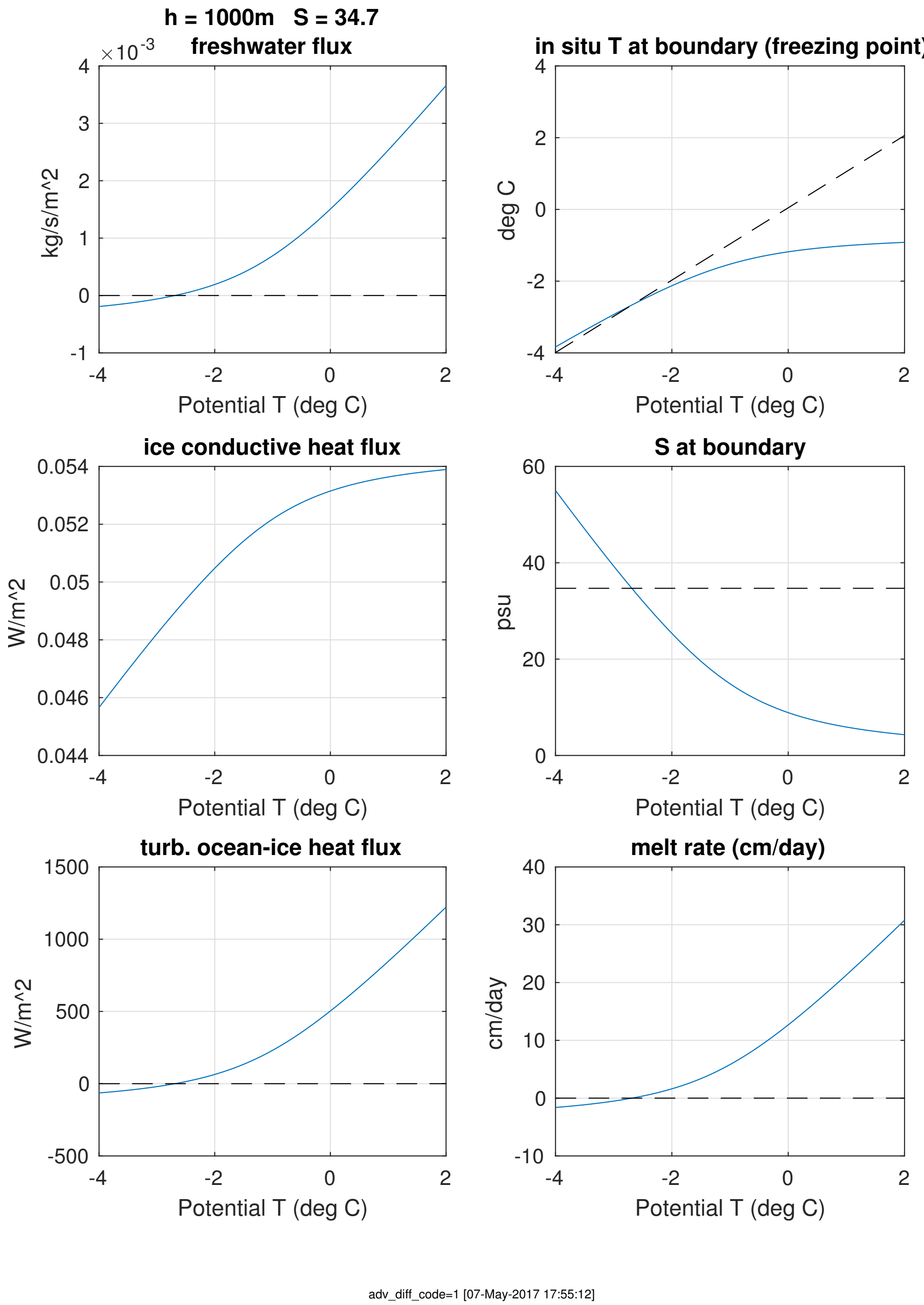

Figure 5: Solution to the Three-Equation model for z = 1000 m, ice conduction with linear ice T profile

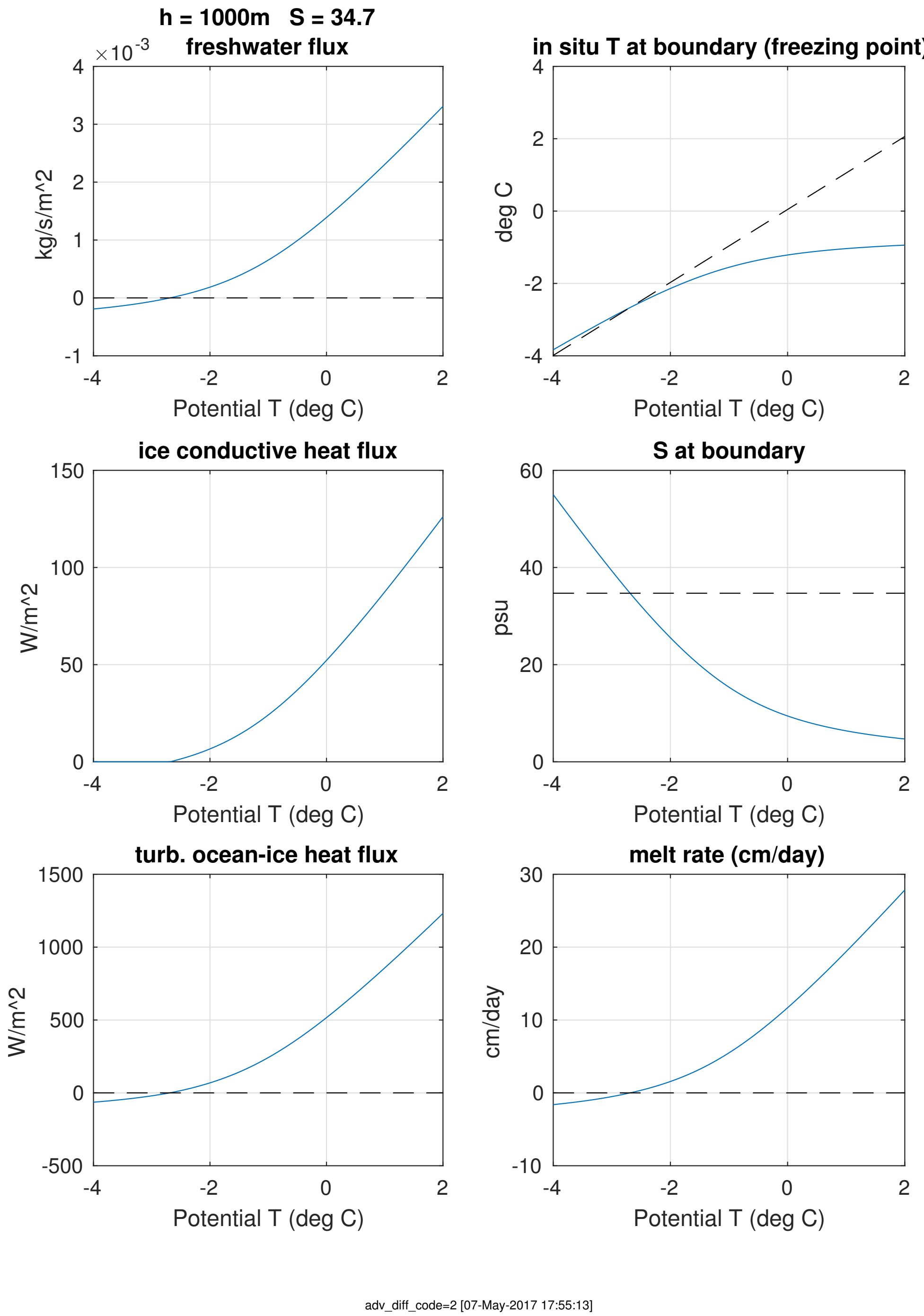

Figure 6: Solution to the Three-Equation model for z = 1000 m, ice conduction with non-linear ice ice T profile

5 Bibliography

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